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On displacement-based theories of sandwich plates with soft core

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Abstract. The paper is concerned with a family of refined models of elastic sandwich plates with soft core. Construction of this family is based upon the kinematic assumptions of Dundrová, Kovařik and Šlapák [4]. The model energy-consistent with this hypothesis turns out to be nonelliptic. However, this model makes it possible to find a generalization of Hoff's [7] model in which transverse normal deformations of the core are partly considered. A proof is given that both this and Hoff's model is correctly stated irrespective of the choice of fields that describe the angles of rotation of the plate cross-sections. On the other hand, in the model of Reissner [18] this flexibility is lost and only one choice of fields standing for rotations is admissible – namely that in which the assumptions of the Lax-Milgram lemma are fulfilled.

1. Introduction

The first successful attempt to describe deformations of sandwich plates with soft core is due to Reissner [18]. According to this concept the equations of transversely homogeneous moderately thick plates [17] can model deformations of the sandwich plates, provided that the bending and shearing stiffnesses are appropriately modified, see also Plantema [14], Ganowicz [6], Kączkowski [8]; for further references see Noor and Burton [13]. The Reissner [18] model has been generalized by Hoff [7] for the case of plates in which the bending energy of face-plates cannot be neglected, see also Wachowiak and Wilde [21], Stamm and Witte [20]. Effects due to bending the core have been considered in Eringen [5]; transverse shearing deformations of the facings have been described by Yu [22]. Special attention should be focussed on the approach of Dundrová, Kovařik and Šlapák [4], suitable for plates with soft core. In this approach, bending of the face-plates as well as transverse normal and transverse shear deformations of the core are taken into account. Moreover, the stresses in the core associated with the assumed kinematics satisfy the equilibrium equations identically, which distinguishes this approach from the others.

In [4] the equilibrium equations of the plate model are derived by conventional averaging of the equilibrium equations across the thickness, which is called a Bollé-Mindlin manner, cf. [19]. In the present paper the construction of the model is based upon the virtual work equation which is the Lax-Milgram equation for the three-dimensional elasticity problem of the plate. According to this method the kinematic hypothesis (DKS) of Dundrová, Kovařik and Šlapák leads to a refined plate model in which the strain energy consists of five components standing for: energy of in-plane deformations, energy of bending of the face-plates, overall bending energy of the sandwich and energy due to transverse shear strains and transverse normal strains of the core. The energy thus defined is positive definite, which assures that the solution is unique, provided it exists. It occurs, however, that for the natural choice of the space V of kinematically admissible fields the bilinear form defining the energy is not V-elliptic. The problem of constructing the space in which the solution would exist remains open.

The proposed sandwich plate model energy-consistent with the (DKS) kinematic hypothesis is a natural departure point for deriving models of simpler form. Upon neglecting the less important terms in the expression standing for the energy of the transverse normal deformations of the core, one arrives at a new refined model of the Hoff type. The remaining terms standing for this energy introduce some slight modifications to the genuine theory of Hoff [7]. It is shown that this amended model is V-elliptic and thus correctly posed. Discarding the terms that stand for the energy of transverse normal deformations leads to the model of Hoff. Neglecting the energy of bending of the face-plates reduces the latter model to the theory of Reissner [18].

One of the aims of this paper is to show that in the models of Hoff type the energy of the facings makes the model stable in the sense that the model is correctly posed irrespective of the choice of kinematic fields allowed within the theory. Moreover, the constant of ellipticity does not depend on the relative thickness of the plate. Neglect of this energy results in the loss of such arbitrariness. Therefore, in the model of Reissner [17, 18] there is only one set of unknowns which assures V-ellipticity of the problem.

The summation convention is used throughout the paper. Greek indices take values 1, 2. Latin indices run over 1, 2, 3.

2. Problem formulation

The subject of consideration is an elastic sandwich plate of constant thickness $2h$, composed of a core layer of thickness $2c$ and of two external layers (face-plates) of thickness d, see Fig. 1. The plate is symmetric with respect to its middle plane Ω. The domain Ω is parametrized by Cartesian coordinates (x_α). The axis $x_3 = z$ is perpendicular to the middle plane. Thus the layers of the plate occupy the following domains:

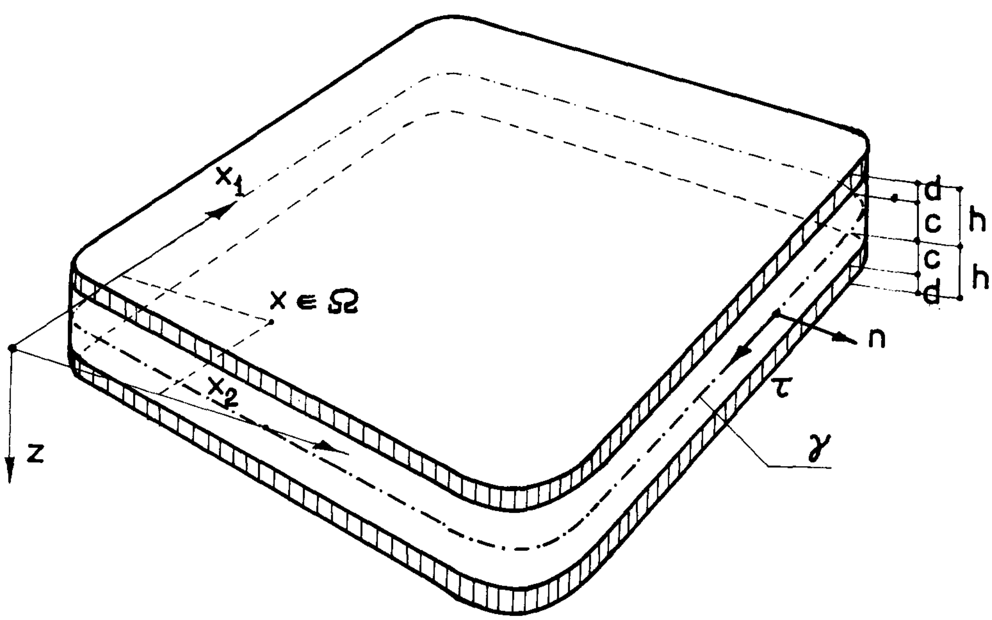

Fig. 1. Geometry of the sandwich plate.

the upper face-plate: $\{\mathbf{x}=(x, z), \quad x=(x_\alpha)\in\Omega, \quad c\leqslant z\leqslant h\}$,

the core: $\{\mathbf{x}=(x, z), \quad x\in\Omega, \quad -c\leqslant z\leqslant c\}$,

the lower face-plate: $\{\mathbf{x}=(x, z), \quad x\in\Omega, \quad -h\leqslant z\leqslant -c\}$,

where $h = c + d$. The middle plane Ω lies on the plane $x_3 = z = 0$. $\Gamma_\pm$ stand for the upper and lower faces, i.e. $\Gamma_\pm = \{\mathbf{x}=(x, \pm h),\ x\in\Omega\}$, and $\Gamma_0 = \{\mathbf{x}=(x, z),\ x\in\gamma=\partial\Omega,\ -h\leqslant z\leqslant h\}$ stands for the lateral cylindrical surface of the plate.

The face-plates are assumed to be made of the same anisotropic linearly elastic material whose moduli are denoted by C^{ijkl}. The planes $z=\text{const}$ are assumed to be planes of material symmetry, hence

$$C^{3\alpha\beta\gamma} = C^{333\alpha} = 0\,. \tag{2.1}$$

The core is assumed to be made of a soft material with moduli C_c^{ijkl}. The in-plane stiffnesses of the core will be neglected. Thus we assume

$$C_c^{\alpha\beta\lambda\mu} = 0, \qquad C_c^{\alpha\beta 33} = 0, \qquad C_c^{\alpha 333} = C_c^{3\beta\gamma\delta} = 0\,. \tag{2.2}$$

The modulus of transverse shear $\mu_c = C_c^{1323}$ and the Young modulus in z-direction $E_c = C_c^{3333}$ are the only non-zero elastic moduli of the core. The face-plates and the core are assumed to be perfectly bonded.

The upper and lower faces $z = \pm h$ are subjected to transverse loadings $p^\pm$. The boundary conditions on the faces $z = \pm h$ read

$$\sigma^{\alpha 3}(x, \pm h) = 0, \qquad \sigma^{33}(x, \pm h) = \pm p^\pm(x)\,, \tag{2.3}$$

where $\sigma^{ij}(x, z)$ stands for stresses at point (x, z). The own weight will be neglected.

We shall confine consideration to the case when the plate is clamped on the lateral surface

$$\Gamma_u = \{(x, z), \quad x\in\gamma_u, \quad -h\leqslant z\leqslant h\}\,,$$

and subject to tractions T^i on the remaining part Γ_σ of the surface Γ_0:

$$\Gamma_\sigma = \{(x, z), \quad x\in\gamma_\sigma, \quad -h\leqslant z\leqslant h\}\,,$$

where $\gamma = \gamma_\sigma \cup \gamma_u$. Let s parametrize the boundary line γ. Let us assume for the sake of simplicity that γ is a smooth curve, i.e. it has no corner points, its curvature being denoted by r. By $n = (n_\alpha)$ and $\tau = (\tau_\alpha)$ we denote the versors outwardly normal and tangent to the curve γ. In the boundary layer of Ω one can define curvilinear coordinates parallel and orthogonal to the line γ. The components n_α and τ_α become the fields determined in this layer. Their derivatives are expressed as follows [10]:

$$n_{\alpha,\beta} = \frac{1}{r}\,\tau_\alpha\tau_\beta, \qquad \tau_{\alpha,\beta} = -\frac{1}{r}\,n_\alpha\tau_\beta\,. \tag{2.4}$$

The in-plane tractions T^α given on the surface Γ_σ are assumed to have the following distribution over the thickness:

$$T^\alpha(s,z)=\begin{cases} p^+_\alpha(s)+z^+r^+_\alpha(s)\,, & c\leq z\leq h\,,\\ 0\,, & -c\leq z\leq c\,,\\ p^-_\alpha(s)+z^-r^-_\alpha(s)\,, & -h\leq z\leq -c\,,\end{cases} \tag{2.5}$$

where $z^\pm = z\mp b$, $b=c+d/2$. The transverse tractions are assumed to be piece-wise constant,

$$T^3=(s,z)=\begin{cases} t_3(s)\,, & c\leq z\leq h\,,\\ t^c_3(s)\,, & -c\leq z\leq c\,,\\ t_3(s)\,, & -h\leq z\leq -c\,.\end{cases} \tag{2.6}$$

3. Stress and kinematic assumptions of Dundrová, Kovařik and Šlapák

In this section we recall the (DKS) assumptions [4] which impose constraints on the stress state and kinematics of the sandwich plate. In contrast to the majority of other approaches, the stresses in the core are assumed to be statically admissible. Moreover the (DKS) approach takes into account both transverse and normal deformations of the core. The face-plates are assumed to be sufficiently thin so that their transverse deformations can be neglected. The drawback of the (DKS) approach is the complicated form of the kinematic constraints, which will be discussed in Sec. 4.

As is typical of plate behaviour modelling, the reduction of the transverse dimension is based upon kinematic and stress assumptions. According to the kinematic assumption the displacements across the face-plates have the form

$$\begin{aligned} u_\alpha(x,z)&=u^0_\alpha\pm b\Big[\gamma_\alpha-\frac{c^2}{3}\,\eta\,\operatorname{div}\boldsymbol{\gamma},_\alpha\Big]-zw,_\alpha\,,\\ u_3(x,z)&=w\,,\end{aligned} \tag{3.1}$$

where $(+)$ refers to $z\in[c,h]$ and $(-)$ refers to $z\in[-h,-c]$; $\eta=\mu_c/E_c$. The fields $\mathbf{u}^0=(u^0_\alpha(x))$, $w=w(x)$, $\boldsymbol{\gamma}=(\gamma_\alpha(x))$ are referred to the middle plane Ω; $\operatorname{div}\boldsymbol{\gamma}=\gamma_\alpha,_\alpha$. The displacements across the thickness of the core are stipulated as below,

$$\begin{aligned} u_\alpha(x,z)&=u^0_\alpha+z\,\frac{b}{c}\Big[\gamma_\alpha-\frac{1}{6}\,\eta(3c^2-z^2)\operatorname{div}\boldsymbol{\gamma},_\alpha\Big]-zw,_\alpha\,,\\ u_3(x,z)&=w+\frac{1}{2}\,\eta\,\frac{b}{c}\,(c^2-z^2)\operatorname{div}\boldsymbol{\gamma}\,.\end{aligned} \tag{3.2}$$

The state of stress within the face-plates is supposed to obey the plane-stress state approximation, i.e.

$$\sigma^{\alpha\beta}=A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{u}),\qquad \sigma^{\alpha3}=\sigma^{33}=0\,, \tag{3.3}$$

where the tensor $\mathbf{A}$ is defined by

$$A^{\alpha\beta\lambda\mu}=C^{\alpha\beta\lambda\mu}-C^{\alpha\beta33}C^{33\lambda\mu}/C^{3333}\,. \tag{3.4}$$

The stresses in the core are assumed in the form compatible with simplifications (2.2),

$$\sigma^{\alpha\beta}=0, \qquad \sigma^{\alpha 3}=2\mu_c\delta^{\alpha\beta}\gamma_{\beta 3}(\mathbf{u}), \qquad \sigma^{33}=E_c\gamma_{33}(\mathbf{u})\,. \tag{3.5}$$

The strains associated with the displacement field $\mathbf{u}$ have been denoted by $\gamma_{ij}=\gamma_{ij}(\mathbf{u})=\frac{1}{2}(u_{i},_{j}+u_{j},_{i})$.

The kinematic hypothesis (3.1, 3.2) assures displacements to be continuous across the thickness. Moreover in each point of the core the equilibrium equations $\sigma^{ij},_{j}=0$ are identically satisfied. To check it one should find strains associated with displacement fields (3.1), (3.2). In the face-plates the deformations read

$$\begin{aligned}&\gamma_{\alpha\beta}(\mathbf{u})=\gamma_{\alpha\beta}(\mathbf{u}^0)\pm b\Big[\gamma_{\alpha\beta}(\boldsymbol{\gamma})+\frac{c^2}{3}\,\eta\kappa_{\alpha\beta}(\operatorname{div}\boldsymbol{\gamma})\Big]+z\kappa_{\alpha\beta}(w)\,,\\ &\gamma_{k3}(\mathbf{u})=0\,,\end{aligned} \tag{3.6}$$

while in the core the strain components are

$$\begin{aligned}&\gamma_{\alpha\beta}(\mathbf{u})=\gamma_{\alpha\beta}(\mathbf{u}^0)+z\,\frac{b}{c}\Big[\gamma_{\alpha\beta}(\boldsymbol{\gamma})+\frac{1}{6}\,(3c^2-z^2)\eta\kappa_{\alpha\beta}(\operatorname{div}\boldsymbol{\gamma})\Big]+z\kappa_{\alpha\beta}(w)\,,\\ &\gamma_{\alpha 3}(\mathbf{u})=\frac{1}{2}\,\frac{b}{c}\,\gamma_\alpha, \qquad \gamma_{33}(\mathbf{u})=-z\,\frac{b}{c}\,\eta\operatorname{div}\boldsymbol{\gamma}\,.\end{aligned} \tag{3.7}$$

We have used the notation $\kappa_{\alpha\beta}(v)=-v,_{\alpha\beta}$.

The stresses can be found with the help of (3.3), (3.5). One can readily see that $\sigma^{\alpha 3},_{\alpha}+\sigma^{33},_{3}=0$ for every $-c\leqslant z\leqslant c$. On the other hand one can check that the distributions of σ^{k3} stresses satisfy neither continuity conditions on the interfaces $z=\pm c$ nor boundary conditions (2.3) on the faces $z=\pm h$. The conditions $\sigma^{\alpha 3}(x,\pm h)=0$ could be fulfilled, if the displacement distributions (3.1) would involve third-order polynomials in z.

4. Energy-consistent approach

The equilibrium of the plate is governed by the equation of virtual work $\delta W=\delta L$, with

$$\begin{aligned}&\delta W=\int_{-h}^{h}\int_{\Omega}\sigma^{ij}\delta\gamma_{ij}\,\mathrm{d}x\,\mathrm{d}z\,,\\ &\delta L=\int_{-h}^{h}\int_{\gamma_\sigma}T^i\delta u_i\,\mathrm{d}s\,\mathrm{d}z+\int_{\Omega}[p^+\delta u_3(x,h)+p^-\delta u_3(x,-h)]\,\mathrm{d}x\,,\end{aligned} \tag{4.1}$$

to hold for every kinematically admissible trial displacement field $\delta\mathbf{u}$. A two-dimensional plate model is called energy-consistent [15], if it is based on a two-dimensional virtual work equation which arises as a consequence of direct substitution of the *a priori* assumed stress-displacement hypotheses into the variational equation (4.1). This section is aimed at deriving the model energy-consistent with the assumptions (3.1)–(3.5). Such a model has not been discussed in the monograph [4], where two-dimensional plate equations have been found in the Bollé-Mindlin manner.

4.1. Basic equations and boundary conditions

Upon considering the assumptions of Sec. 3 one can reduce the virtual work of stresses to the form

$$\delta W = \int_{\Omega} (N^{\alpha\beta}\delta\mathring{\gamma}_{\alpha\beta} + M^{\alpha\beta}\delta k_{\alpha\beta} + \mathfrak{M}^{\alpha\beta}\delta\mu_{\alpha\beta} + G^{\alpha\beta}\delta g_{\alpha\beta} + Q^{\alpha}\delta\gamma_{\alpha})\,\mathrm{d}x\,, \tag{4.2}$$

where

$$\delta\mathring{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta}(\delta\mathbf{u}^0), \qquad \delta k_{\alpha\beta} = \kappa_{\alpha\beta}(\delta w), \qquad \delta\mu_{\alpha\beta} = \gamma_{\alpha\beta}(\delta\boldsymbol{\gamma}), \qquad \delta g_{\alpha\beta} = \kappa_{\alpha\beta}(\operatorname{div}\delta\boldsymbol{\gamma})\,. \tag{4.3}$$

The stress resultants involved in (4.2) are given by

$$N^{\alpha\beta} = N_{+}^{\alpha\beta} + N_{-}^{\alpha\beta}, \qquad M^{\alpha\beta} = m_{+}^{\alpha\beta} + m_{-}^{\alpha\beta} + M_{f}^{\alpha\beta}\,,$$

$$\mathfrak{M}^{\alpha\beta} = M_{f}^{\alpha\beta} \underline{- \eta\,\frac{b}{c}\,S\delta^{\alpha\beta}}, \qquad Q^{\alpha} = \frac{b}{c}\,\hat{Q}^{\alpha}, \qquad G^{\alpha\beta} = \frac{1}{3}\,\eta c^2 M_{f}^{\alpha\beta}\,.$$

where

$$N_{+}^{\alpha\beta} = \int_{c}^{h} \sigma^{\alpha\beta}\,\mathrm{d}z, \qquad N_{-}^{\alpha\beta} = \int_{-h}^{-c} \sigma^{\alpha\beta}\,\mathrm{d}z, \qquad m_{\pm}^{\alpha\beta} = \int_{-d/2}^{d/2} (z \mp b)\sigma^{\alpha\beta}\,\mathrm{d}z,$$

$$M_{f}^{\alpha\beta} = b(N_{+}^{\alpha\beta} - N_{-}^{\alpha\beta}), \qquad \hat{Q}^{\alpha} = \int_{-c}^{c} \sigma^{\alpha 3}\,\mathrm{d}z, \qquad S = \int_{-c}^{c} z\sigma^{33}\,\mathrm{d}z\,.$$

The quantities $N_{\pm}^{\alpha\beta}$, $m_{\pm}^{\alpha\beta}$ are stress and couple resultants referred to the middle planes of the faces, respectively; $\hat{Q}^{\alpha}$ are transverse forces in the core and S describes the resultant of σ^{33} stresses that rend the sandwich in the transverse direction. $M_{f}^{\alpha\beta}$ represent moments of in-plane forces of the faces, referred to the $z = 0$ plane.

Therefore, by their definition, the quantities $N^{\alpha\beta}$ are conventional resultants of the in-plane stresses, $M^{\alpha\beta}$ represent the total moments referred to the middle plane and Q^{α} are transverse forces corrected with the coefficient b/c. The term underlined in the definition of $\mathfrak{M}^{\alpha\beta}$ is a consequence of taking into account the transverse stretching of the core. $G^{\alpha\beta}$ are extra moments due to the nonlinear distribution of stresses $\sigma^{\alpha\beta}$ across the plate faces.

The stress resultants **N**, **M**, $\mathfrak{M}$, **G**, **Q** are interrelated with the strain fields $\mathring{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{u}^0)$, $k_{\alpha\beta} = \kappa_{\alpha\beta}(w)$, (γ_β), $\mu_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\gamma})$, $g_{\alpha\beta} = \kappa_{\alpha\beta}(\operatorname{div}\boldsymbol{\gamma})$ according to the formulae

$$\begin{aligned}
N^{\alpha\beta} &= 2dA^{\alpha\beta\lambda\mu}\mathring{\gamma}_{\lambda\mu}\,,\\
M^{\alpha\beta} &= \left(\frac{1}{6}\,d^3 + 2db^2\right)A^{\alpha\beta\lambda\mu}k_{\lambda\mu} + 2db^2A^{\alpha\beta\lambda\mu}\left(\mu_{\lambda\mu} + \frac{1}{3}\,\eta c^2 g_{\lambda\mu}\right),\\
\mathfrak{M}^{\alpha\beta} &= \left(2db^2A^{\alpha\beta\lambda\mu} + \frac{2}{3}\,\eta c b^2\mu_c\delta^{\alpha\beta}\delta^{\lambda\mu}\right)\mu_{\lambda\mu} + 2db^2A^{\alpha\beta\lambda\mu}\left(k_{\lambda\mu} + \frac{1}{3}\,\eta c^2 g_{\lambda\mu}\right),\\
G^{\alpha\beta} &= \frac{2}{3}\,\eta d c^2 b^2 A^{\alpha\beta\lambda\mu}\left(k_{\lambda\mu} + \mu_{\lambda\mu} + \frac{1}{3}\,\eta c^2 g_{\lambda\mu}\right),\\
Q^{\alpha} &= \frac{2}{c}\,b^2\mu_c\delta^{\alpha\beta}\gamma_\beta\,.
\end{aligned} \tag{4.4}$$

The relation between Q^α and γ_β follows directly from the kinematic assumptions. To improve accuracy of this relationship one can introduce a shear correction factor, cf. Mindlin [11].

Let us explain now why the shearing stiffness turned out to be $2b^2\mu_c/c$ and not $2c\mu_c$. According to the relation $(3.7)_2$ the resultant shear β_α of the core reads $\beta_\alpha = 2\gamma_{\alpha 3} = (b/c)\gamma_\alpha$. This equation can be rewritten in the form

$$\beta_\alpha = \chi_\alpha + \gamma_\alpha , \qquad \chi_\alpha = \frac{d}{2c}\gamma_\alpha . \tag{4.5}$$

The above relations are illustrated in Fig. 2 where we have assumed that $\mathbf{u}^0 = \mathbf{0}$, $w = 0$, $\gamma_\alpha = \text{const}$. Note that the equation $c\chi_\alpha = (d/2)\gamma_\alpha$ follows from a geometrical consideration. The following identity holds,

$$\hat{Q}^\alpha \delta\beta_\alpha = Q^\alpha \delta\gamma_\alpha , \tag{4.6}$$

where $\hat{Q}^\alpha = 2c\mu_c\delta^{\alpha\sigma}\beta_\sigma$. Of the two possibilities β_α and γ_α, it will appear that the latter unknowns are more convenient.

Let us pass now to evaluating the virtual work of external loading. The work of transverse loading $p^\pm$ equals

$$\delta L_1 = \int_\Omega p\,\delta w\,\mathrm{d}x , \qquad p = p^+ + p^- . \tag{4.7}$$

After integration over the thickness and integration by parts with the help of the formulae (2.4), the virtual work of the tractions T^i applied to the boundary Γ_σ can be cast in the form

$$\delta L_2 = \int_{\gamma_\sigma} (\tilde{N}_n \delta u_n^0 + \tilde{N}_\tau \delta u_\tau^0 + \tilde{Q}\delta w - \tilde{M}_n \delta w_{,n} + \tilde{G}_n \delta\gamma_{n,nn} + \bar{G}_n \delta\gamma_{n,n} + \bar{G}_\tau \delta\gamma_{\tau,n}$$
$$+ \hat{G}_n \delta\gamma_n + \hat{G}_\tau \delta\gamma_\tau)\, ds . \tag{4.8}$$

The effective boundary forces and moments are

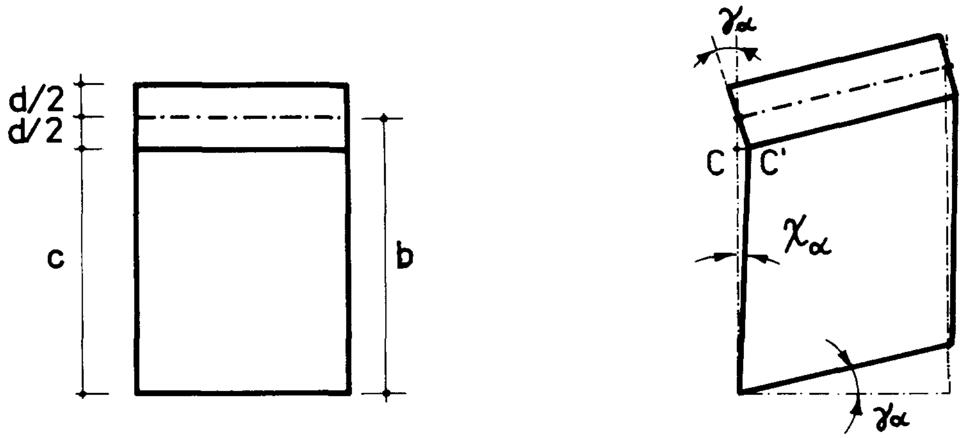

Fig. 2. Deformations due to shear of the core. Note that $CC' = c\chi_\alpha$ and $CC' = (d/2)\gamma_\alpha$, hence $\chi_\alpha = d(2c)^{-1}\gamma_\alpha$.

$$
\begin{aligned}
&\tilde{N}_n = \tilde{N}_\alpha n_\alpha, \qquad \tilde{N}_\tau = \tilde{N}_\alpha \tau_\alpha, \qquad \tilde{N}_\alpha = d(p_\alpha^+ + p_\alpha^-), \\
&\tilde{Q} = Q + \tilde{M}_{\tau},_{\tau}, \qquad Q = 2(dt_3 + ct_3^c), \\
&\tilde{M}_\tau = \tilde{M}_\alpha \tau_\alpha, \qquad \tilde{M}_\alpha = db(p_\alpha^+ - p_\alpha^-) + \frac{d^3}{12}(r_\alpha^+ + r_\alpha^-), \\
&\tilde{M}_n = \tilde{M}_\alpha n_\alpha, \qquad \tilde{G}_n = \tilde{G}_\alpha n_\alpha, \qquad \tilde{G}_\alpha = \frac{d}{3} bc^2\eta(p_\alpha^- - p_\alpha^+), \\
&\bar{G}_n = \frac{1}{r}\tilde{G}_n - \tilde{G}_{\tau},_{\tau} + \tilde{G}, \\
&\tilde{G}_\tau = \tilde{G}_\alpha \tau_\alpha, \qquad \tilde{G} = \frac{2}{3}\eta c^2 b t_3^c, \qquad \bar{G}_\tau = -\tilde{G}_{n},_{\tau}, \\
&\hat{G}_\tau = \tilde{\mathfrak{M}}_\tau + \tilde{G}_{\tau},_{\tau\tau} - \left(\frac{1}{r}\tilde{G}_n\right),_{\tau} - \tilde{G},_{\tau}, \\
&\tilde{\mathfrak{M}}_\tau = \tilde{\mathfrak{M}}_\alpha \tau_\alpha, \qquad \tilde{\mathfrak{M}}_\alpha = db(p_\alpha^+ - p_\alpha^-), \\
&\hat{G}_n = -\frac{1}{r}\tilde{G}_{\tau},_{\tau} + \tilde{\mathfrak{M}}_n + \frac{1}{r}\tilde{G}, \qquad \tilde{\mathfrak{M}}_n = \tilde{\mathfrak{M}}_\alpha n_\alpha .
\end{aligned}
\tag{4.9}
$$

Having found the expressions (4.2) and (4.8) one can arrive at the equilibrium equations

$$
\begin{aligned}
&-N^{\alpha\beta},_{\beta} = 0, \qquad -M^{\alpha\beta},_{\alpha\beta} = p, \\
&-\mathfrak{M}^{\alpha\beta},_{\beta} + \delta^{\alpha\beta}G^{\lambda\mu},_{\lambda\mu\beta} + Q^\alpha = 0,
\end{aligned}
\tag{4.10}
$$

to be satisfied in Ω and natural boundary conditions along the arc γ_σ,

$$
\begin{aligned}
&N_n = \tilde{N}_n, \qquad N_\tau = \tilde{N}_\tau, \qquad M^{\alpha\beta},_{\beta} n_\alpha + M_{\tau},_{\tau} = \tilde{Q}, \qquad M_n = \tilde{M}_n, \\
&G_n = \tilde{G}_n, \qquad G - G_{\tau},_{\tau} + \frac{1}{r}G_n = \bar{G}_n, \qquad G_{n},_{\tau} = -\bar{G}_\tau, \\
&\mathfrak{M}_n - \frac{1}{r}G_{\tau},_{\tau} + \frac{1}{r}G - \hat{G} = \hat{G}_n, \\
&\mathfrak{M}_\tau - \left(\frac{1}{r}G_n\right),_{\tau} + G_{\tau},_{\tau\tau} - G,_{\tau} = \hat{G}_\tau,
\end{aligned}
\tag{4.11}
$$

where

$$
\begin{aligned}
&(N_n, M_n, \mathfrak{M}_n, G_n) = (N^{\alpha\beta}, M^{\alpha\beta}, \mathfrak{M}^{\alpha\beta}, G^{\alpha\beta})n_\alpha n_\beta, \\
&(N_\tau, M_\tau, \mathfrak{M}_\tau, G_\tau) = (N^{\alpha\beta}, M^{\alpha\beta}, \mathfrak{M}^{\alpha\beta}, G^{\alpha\beta})n_\alpha \tau_\beta, \\
&G = G^{\alpha\beta},_{\beta} n_\alpha, \qquad \hat{G} = G^{\alpha\beta},_{\alpha\beta} .
\end{aligned}
\tag{4.12}
$$

One can note that the above problem of finding $(\mathbf{u}^0, \boldsymbol{\gamma}, w)$ splits up into two independent problems: the conventional plane-stress problem of finding the field $\mathbf{u}^0$ and the bending problem of finding $(\boldsymbol{\gamma}, w)$.

Equations of equilibrium can be expressed in terms of displacement fields. In the general case their form is complicated and will not be reported. Let us display them in the case when the face-plates are isotropic, i.e. when

$$A^{\alpha\beta\lambda\mu} = \mu\left[(\delta^{\alpha\lambda}\delta^{\beta\mu} + \delta^{\alpha\mu}\delta^{\beta\lambda}) + \frac{2\nu}{1-\nu}\,\delta^{\alpha\beta}\delta^{\lambda\mu}\right],$$

where μ stands for the shear modulus and ν for the Poisson ratio of the facings.

The in-plane equations read

$$\Delta u^0_\alpha + \frac{1+\nu}{1-\nu}\, u^0_{\sigma},_{\sigma\alpha} = 0\,. \tag{4.13}$$

The equations of bending have the form

$$\begin{aligned}
&-\frac{4\mu db^2}{1-\nu}\,\Delta\Big(w,_\alpha - \underline{\tfrac{1}{3}\,\eta c^2\Delta w,_\alpha}\Big) + 2db^2\mu\Delta\gamma_\alpha \\
&+\left[2d\mu\,\frac{1+\nu}{1-\nu}\,b^2 + \frac{2}{3}\,\eta\mu_c cb^2\right]\operatorname{div}\boldsymbol{\gamma},_\alpha - \frac{2}{c}\,b^2\mu_c\gamma_\alpha \\
&\underline{-\frac{8}{3}\,\frac{d\mu\eta}{1-\nu}\,(cb)^2\Delta\operatorname{div}\boldsymbol{\gamma},_\alpha} + \underline{\frac{4}{9}\,\mathrm{d}\,\frac{\mu}{1-\nu}\,\eta^2c^4b^2\Delta^2\operatorname{div}\boldsymbol{\gamma},_\alpha} = 0\,, \\
&\frac{\mu}{1-\nu}\left(\frac{\mathrm{d}^3}{3} + 4db^2\right)\Delta^2 w - \frac{4\mu db^2}{1-\nu}\,\Delta\Big(\operatorname{div}\boldsymbol{\gamma} - \underline{\tfrac{1}{3}\,\eta c^2\,\Delta\operatorname{div}\boldsymbol{\gamma}}\Big) = p\,.
\end{aligned} \tag{4.14}$$

Let us define the function $\chi = \gamma_1,_2 - \gamma_2,_1$. One can check that this function satisfies the following equation,

$$\Delta\chi - \frac{10}{\bar{h}^2}\,\chi = 0\,, \tag{4.15}$$

where

$$\bar{h} = (10cd\mu/\mu_c)^{1/2}\,. \tag{4.16}$$

The equation (4.15) is a counterpart of the boundary-layer equation of the theory of plates with moderate thickness [8, 9]; $\bar{h}$ occurs to be an average thickness of the plate.

4.2. Formulation of boundary value problems

Let Ω be a domain with a Lipschitz boundary γ, cf. [12], without corner points. For the sake of simplicity let us assume that the plate is clamped along $\Gamma_u = \Gamma_0$, i.e., $\gamma_u = \gamma$. The space of kinematically admissible fields $\mathbf{u}^0$, $\boldsymbol{\gamma}$, w can be defined as below

$$V(\Omega) = [H^1_0(\Omega)]^2 \times [H^3_0(\Omega)]^2 \times H^2_0(\Omega)\,,$$

where $H^k_0(\Omega)$ is a Sobolev space of functions whose traces and traces of their derivatives $\partial^l/\partial n^l$, $l = 1, \ldots, k-1$, vanish on γ. Moreover let us assume that there exist positive constants k_α such that

$$A^{\alpha\beta\lambda\mu}a_{\alpha\beta}a_{\lambda\mu} \leqslant k_1 a_{\alpha\beta}a_{\alpha\beta}\,, \qquad A^{\alpha\beta\lambda\mu}a_{\alpha\beta}a_{\lambda\mu} \geqslant k_2 a_{\alpha\beta}a_{\alpha\beta} \tag{4.17}$$

for every $\mathbf{a} = (a_{\alpha\beta}) \in M_s^2$; M_s^2 stands for the space of real symmetric 2×2 matrices. The remaining moduli η, μ_c are positive. Moreover we assume that $p \in L^2(\Omega)$.

The virtual work of stress resultants (4.2) can be rearranged into the bilinear form

$$
\begin{aligned}
&a(\mathbf{u}^0, \boldsymbol{\gamma}, w; \mathbf{v}^0, \boldsymbol{\psi}, v) \\
&= \int_\Omega \Big\{ 2dA^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{u}^0)\gamma_{\lambda\mu}(\mathbf{v}^0) + 2\,\frac{d^3}{12}\,A^{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(w)\kappa_{\lambda\mu}(v) \\
&\quad + 2\,\frac{b^2}{c}\,\mu_c\gamma_\alpha\psi_\alpha + \frac{2}{3}\,cb^2\eta\mu_c \operatorname{div}\boldsymbol{\gamma} \operatorname{div}\boldsymbol{\psi} \\
&\quad + 2db^2A^{\alpha\beta\lambda\mu}\Big[\kappa_{\alpha\beta}(w) + \gamma_{\alpha\beta}(\boldsymbol{\gamma}) - \underline{\tfrac{1}{3}\,\eta c^2\gamma_{\sigma},_{\sigma\alpha\beta}}\Big]\Big[\kappa_{\lambda\mu}(v) + \gamma_{\lambda\mu}(\boldsymbol{\psi}) - \underline{\tfrac{1}{3}\,\eta c^2\psi_{\gamma},_{\gamma\lambda\mu}}\Big]\Big\}\,\mathrm{d}x
\end{aligned}
\tag{4.18}
$$

where $\mathbf{v}^0 = \delta\mathbf{u}^0$, $\boldsymbol{\psi} = \delta\boldsymbol{\gamma}$, $v = \delta w$. One can see that the bilinear form a: $V(\Omega) \times V(\Omega) \to R$ is symmetric, and, by virtue of the estimate $(4.17)_2$, is positive definite.

The virtual work of external loading can be expressed by the linear form f: $V(\Omega) \to R$ defined by

$$
f(v) = \int_\Omega pv\,\mathrm{d}x\,. \tag{4.19}
$$

The boundary-value problem of the clamped plate reads

$$
(\mathscr{P}_1)\begin{cases} \text{find } (\mathbf{u}^0, \boldsymbol{\gamma}, w) \in V(\Omega) \quad \text{such that} \\ a(\mathbf{u}^0, \boldsymbol{\gamma}, w; \mathbf{v}^0, \boldsymbol{\psi}, v) = f(v) \quad \text{for every } (\mathbf{v}^0, \boldsymbol{\psi}, v) \in V(\Omega)\,. \end{cases} \tag{4.20}
$$

By standard arguments we conclude that $\mathbf{u}^0 = \mathbf{0}$. Moreover, one can easily prove that the solution $(\boldsymbol{\gamma}, w)$ is unique. However, we shall prove below that the bilinear form $a(.;.)$ is not $V(\Omega)$-elliptic, thus the assumptions of the Lax-Milgram lemma are not satisfied and the problem of whether the solution $(\boldsymbol{\gamma}, w)$ exists will remain open. To prove that the form (4.18) is not $V(\Omega)$-elliptic let us start with

LEMMA 1. *Let Ω satisfy the above-mentioned conditions. For every $M > 0$ there exists $\Phi \in H_0^4(\Omega)$ such that*

$$
\|\nabla\Phi\|^2_{[H_0^3(\Omega)]^2} > M\|\nabla\Phi\|^2_{[H_0^1(\Omega)]^2}\,. \tag{4.21}
$$

The proof of the lemma is standard and will be omitted.

To prove that the bilinear form $a(.:.)$ is not $V(\Omega)$-elliptic, it is sufficient to show that for every constant $c_1 > 0$ there exists $\mathbf{z} = (\mathbf{u}^0, \boldsymbol{\gamma}, w) \in V(\Omega)$ such that

$$
a(\mathbf{z}; \mathbf{z}) < c_1\|\mathbf{z}\|^2_{V(\Omega)}\,. \tag{4.22}
$$

Let us fix $c_1 > 0$ and take $\mathbf{z} = (\mathbf{0}, \boldsymbol{\gamma}, 0)$, where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$, $\gamma_1 = \Phi,_2$, $\gamma_2 = -\Phi,_1$; $\Phi \in H_0^4(\Omega)$. Since $\operatorname{div}\boldsymbol{\gamma} = 0$ one gets

$$
a(\mathbf{z}; \mathbf{z}) = 2b^2\int_\Omega \Big[dA^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\boldsymbol{\gamma})\gamma_{\lambda\mu}(\boldsymbol{\gamma}) + \frac{1}{c}\,\mu_c\gamma_\alpha\gamma_\alpha\Big]\,\mathrm{d}x\,. \tag{4.23}
$$

With the help of the relation $(4.17)_1$ one can fix a constant $k_3 > 0$ such that

$$a(\mathbf{z}; \mathbf{z}) \leq k_3 \|\boldsymbol{\gamma}\|^2_{[H^1_0(\Omega)]^2} = k_3 \|\nabla\Phi\|^2_{[H^1_0(\Omega)]^2} . \tag{4.24}$$

According to Lemma 1, for a constant $M = k_3/c_1$ one can choose a function $\Phi \in H^4_0(\Omega)$ such that

$$a(\mathbf{z}; \mathbf{z}) < \frac{k_3}{M} \|\nabla\Phi\|^2_{[H^3_0(\Omega)]^2} = c\|\mathbf{z}\|^2_{V(\Omega)} . \tag{4.25}$$

Thus the bilinear form $a(.;.)$ is not $V(\Omega)$-elliptic, which suggests that the solution to problem $\mathcal{P}_1$ should be looked for in another space. The existence problem remains open.

5. Generalized model of Hoff

In this section we derive and discuss a generalization of the Hoff's [7] model of sandwich plates. Apart from the effect of bending of face-plates considered in the original paper by Hoff [7], the proposed description involves an effect of transverse normal deformation γ_{33} of the core.

5.1. Formulation

Let the maximum eigenvalue of the strain tensors $\mu_{\lambda\mu}$ and $k_{\lambda\mu}$ be denoted by m and k, respectively. Let L stand for the wave length of the deformation pattern associated with deformations $\mu_{\lambda\mu}$. Hence $\mu_{\lambda\mu} = O(m)$, $\mu_{\lambda\mu},_\alpha = O(m/L)$, $\mu_{\lambda\mu},_{\alpha\beta} = O(m/L^2)$.

The model to be put forward is based upon the following assumptions:

$$(c/L)^2 \ll 1, \qquad (c/L)^2 \ll k/m . \tag{5.1}$$

According to them one can make the following simplifications,

$$\begin{aligned} \mu_{\lambda\mu} + \frac{1}{3}\eta c^2 g_{\lambda\mu} &= \left[1 + \frac{1}{3}\eta\left(\frac{c}{L}\right)^2\right] O(m) = O(m) , \\ k_{\lambda\mu} + \frac{1}{3}\eta c^2 g_{\lambda\mu} &= \left[1 + \eta\frac{m}{3k}\left(\frac{c}{L}\right)^2\right] O(k) = O(k) . \end{aligned} \tag{5.2}$$

Thus according to the assumptions (5.1) the tensor $g_{\alpha\beta}$ can be neglected in the constitutive relations (4.4) and in the expression that defines the strain energy. The virtual work of stresses takes the form

$$\delta W = \int_\Omega (N^{\alpha\beta}\delta\mathring{\gamma}_{\alpha\beta} + M^{\alpha\beta}\delta k_{\alpha\beta} + \mathfrak{M}^{\alpha\beta}\delta\mu_{\alpha\beta} + Q^\alpha\delta\gamma_\alpha)\,\mathrm{d}x , \tag{5.3}$$

where $N^{\alpha\beta}$, Q^α are given by $(4.4)_1$, $(4.4)_5$ and

$$\begin{aligned} M^{\alpha\beta} &= \left(\frac{d^3}{6} + 2db^2\right)A^{\alpha\beta\lambda\mu}k_{\lambda\mu} + 2db^2A^{\alpha\beta\lambda\mu}\mu_{\lambda\mu} , \\ \mathfrak{M}^{\alpha\beta} &= 2db^2A^{\alpha\beta\lambda\mu}k_{\lambda\mu} + \left(2db^2A^{\alpha\beta\lambda\mu} + \frac{2}{3}\eta cb^2\delta^{\alpha\beta}\delta^{\lambda\mu}\right)\mu_{\lambda\mu} . \end{aligned} \tag{5.4}$$

Similarly one should simplify the virtual work of boundary forces T^i. To make the expression for δL_2 compatible with formula (5.3), we neglect the terms depending on η. Hence we obtain

$$\delta L = \int_\Omega p\,\delta w\,\mathrm{d}x + \int_{\gamma_\sigma} (\tilde{N}_n \delta u_n^0 + \tilde{N}_\tau \delta u_\tau^0 + \tilde{Q}\delta w - \tilde{M}_n \delta w_{,n} + \tilde{\mathfrak{M}}_n \delta\gamma_n + \tilde{\mathfrak{M}}_\tau \delta\gamma_\tau)\,ds\,. \tag{5.5}$$

The equilibrium equations have the form (4.10) with the term $G^{\lambda\mu}{}_{,\lambda\mu\alpha}$ omitted. The boundary conditions along γ_σ are

$$\begin{aligned} &N_n = \tilde{N}_n, \qquad N_\tau = \tilde{N}_\tau, \qquad M^{\alpha\beta}{}_{,\beta} n_\alpha + M_{\tau,\tau} = \tilde{Q}\,, \\ &M_n = \tilde{M}_n, \qquad \mathfrak{M}_n = \tilde{\mathfrak{M}}_n, \qquad \mathfrak{M}_\tau = \tilde{\mathfrak{M}}_\tau\,. \end{aligned} \tag{5.6}$$

As previously, in the case of face-plates being isotropic, the equilibrium equations can be expressed in terms of kinematic fields. They assume the form of equations (4.13), (4.14) with the underscored terms omitted. The boundary-layer equation (4.15) remains unchanged.

5.2. Boundary-value problems

As in Sec. 4.2 let us assume that the plate is clamped along its boundary. The space of kinematically admissible fields $\mathbf{u}^0$, $\boldsymbol{\gamma}$, w is here defined by

$$W(\Omega) = [H_0^1(\Omega)]^2 \times [H_0^1(\Omega)]^2 \times H_0^2(\Omega)\,.$$

The conditions (4.17) and the assumptions concerning regularity properties of the domain Ω and of the loading p are preserved. The bilinear form associated with the expression (5.3) will be denoted by $a_{GH}(.\,,.\,,.;\,.\,,.\,,.)$; a_{GH}: $W(\Omega)\times W(\Omega)\to R$ is defined by the formula (4.18) with the underlined terms omitted. The form $a_{GH}(.;.)$ is still symmetric and positive definite. Moreover, one can prove that this form is $W(\Omega)$-elliptic. Since this proof is similar to the proof of ellipticity of the bilinear form of the Love's first-order linear shell theory [1], only an outline of the proof will be reported.

By virtue of the positive definiteness condition $(4.17)_2$ one can estimate

$$\begin{aligned} a_{GH}(\mathbf{u}^0, \boldsymbol{\gamma}, w; \mathbf{u}^0, \boldsymbol{\gamma}, w) \geqslant{}& 2dk_2\|\gamma_{\alpha\beta}(\mathbf{u}^0)\gamma_{\alpha\beta}(\mathbf{u}^0)\|^2 + k_2\frac{d^3}{6}\|\kappa_{\alpha\beta}(w)\kappa_{\alpha\beta}(w)\|^2 \\ &+2db^2k_2\|\rho_{\alpha\beta}\rho_{\alpha\beta}\|^2 + 2\frac{b^2}{c}\mu_c\|\gamma_\alpha\gamma_\alpha\|^2\,, \end{aligned} \tag{5.7}$$

where $\|\cdot\|$ stands for the norm in $L^2(\Omega)$ and $\rho_{\alpha\beta} = \kappa_{\alpha\beta}(w) + \gamma_{\alpha\beta}(\boldsymbol{\gamma})$. Now we can estimate

$$\|\rho_{\alpha\beta}\rho_{\alpha\beta}\|^2 \geqslant \frac{\beta}{1+\beta}\|\gamma_{\alpha\beta}(\boldsymbol{\gamma})\gamma_{\alpha\beta}(\boldsymbol{\gamma})\|^2 - \beta\|\kappa_{\alpha\beta}(w)\kappa_{\alpha\beta}(w)\|^2\,, \tag{5.8}$$

where $\beta > 0$, and choose β such that $\beta < \frac{1}{12}(d/b)^2$. Thus there exists a positive constant k_4 such that

$$a_{GH}(\mathbf{z};\mathbf{z}) \geqslant k_4[\|\gamma_{\alpha\beta}(\mathbf{u}^0)\gamma_{\alpha\beta}(\mathbf{u}^0)\|^2 + \|\gamma_\alpha\gamma_\alpha\|^2 + \|\kappa_{\alpha\beta}(w)\kappa_{\alpha\beta}(w)\|^2 + \|\gamma_{\alpha\beta}(\boldsymbol{\gamma})\gamma_{\alpha\beta}(\boldsymbol{\gamma})\|^2]\,, \tag{5.9}$$

where $\mathbf{z} = (\mathbf{u}^0, \boldsymbol{\gamma}, w)$. For simplicity the differences in dimensions of the terms at the right-hand side of (5.9) have been neglected. By applying the Korn inequalities in a standard manner [1, 12] the $W(\Omega)$-ellipticity of the bilinear form $a_{GH}(.;.)$ follows. Furthermore one can show that this form is continuous. Thus according to the Lax-Milgram lemma the problem

$$(\mathscr{P}_2)\begin{cases} \text{find } \mathbf{z} = (\mathbf{u}^0, \boldsymbol{\gamma}, w) \in W(\Omega) \quad \text{such that} \\ a_{GH}(\mathbf{z}; \mathbf{v}) = f(v) \quad \text{for every } \mathbf{v} = (\mathbf{v}^0, \boldsymbol{\psi}, v) \in W(\Omega) \end{cases} \tag{5.10}$$

is well-posed; its solution exists and is unique.

6. The model due to Hoff

In this section a model will be discussed that follows from the one derived in Sec. 5 by neglecting the transverse normal deformations of the core. It will be proved that this model, coinciding with the model of Hoff [7], cf. [20, 21], is well-posed.

6.1. *Formulation in terms of $\boldsymbol{u}^0$, $\boldsymbol{\gamma}$, w*

In the kinematic assumptions (3.1), (3.2), we take into account the transverse normal deformability of the core. To derive the model in which this effect is neglected one should assume that $E_c \gg \mu_c$, viz. one should substitute $\eta = \mu_c/E_c \approx 0$ into the definitions of δW and δL. Then the virtual work of stresses assumes the form (5.3) in which the stress resultants $N^{\alpha\beta}$, Q^α, $M^{\alpha\beta}$ are expressed by $(4.4)_1$, $(4.4)_5$ and $(5.4)_1$, respectively, while

$$\mathfrak{M}^{\alpha\beta} = 2db^2 A^{\alpha\beta\lambda\mu}(k_{\lambda\mu} + \mu_{\lambda\mu}) . \tag{6.1}$$

The virtual work of external loading is given by the formula (5.5). Also the boundary conditions (5.6) preserve their form. Moreover, it is readily seen that the bilinear form associated with the present formula for δW is $W(\Omega)$-elliptic; the proof given in Sec. 5.2 holds good in the present case. Thus the model of Hoff is well-posed.

6.2. *Introduction of the total angles of rotation*

The angles γ_α do not stand for the total angles of rotation of fibres which lie on the plate middle plane. The total angles of rotation equal $\varphi_\alpha = -w,_\alpha + \gamma_\alpha$. The deformations $\gamma_{\alpha\beta}(\boldsymbol{\varphi})$ associated with the field $\boldsymbol{\varphi}$ will be denoted by $\rho_{\alpha\beta}$. Then

$$\rho_{\alpha\beta} = k_{\alpha\beta} + \mu_{\alpha\beta} . \tag{6.2}$$

Let us introduce the moments of stresses in external layers: $\mathscr{M}^{\alpha\beta} = M^{\alpha\beta} - \mathfrak{M}^{\alpha\beta}$. The virtual work (5.3) assumes the form

$$\delta W = \int_\Omega [N^{\alpha\beta}\delta\mathring{\gamma}_{\alpha\beta} + \underline{\mathscr{M}^{\alpha\beta}\delta k_{\alpha\beta}} + \mathfrak{M}^{\alpha\beta}\delta\rho_{\alpha\beta} + Q^\alpha(\delta\varphi_\alpha + \delta w,_\alpha)]\,\mathrm{d}x \tag{6.3}$$

where $N^{\alpha\beta}$ is given by $(4.4)_1$ and

$$\mathcal{M}^{\alpha\beta} = 2\,\frac{d^3}{12}\,A^{\alpha\beta\lambda\mu}k_{\lambda\mu}\,, \qquad \mathfrak{M}^{\alpha\beta} = 2db^2A^{\alpha\beta\lambda\mu}\rho_{\lambda\mu}\,,$$
$$Q^{\alpha} = \frac{2}{c}\,b^2\mu_c(w,_{\alpha} + \varphi_{\alpha})\,. \tag{6.4}$$

Similarly one can rearrange the virtual work of external loadings. Since the boundary line γ has no corner points the expression (5.5) takes the form

$$\delta L = \int_{\Omega} p\delta w \,\mathrm{d}x + \int_{\gamma_\sigma} [\tilde{N}_n\delta u_n^0 + \tilde{N}_\tau\delta u_\tau^0 + (Q + \underline{\tilde{\mathcal{M}}_{\tau},_{\tau}})\delta w - \underline{\tilde{\mathcal{M}}_n\delta w,_n} + \tilde{\mathfrak{M}}_n\delta\varphi_n + \tilde{\mathfrak{M}}_\tau\delta\varphi_\tau]\,ds\,, \tag{6.5}$$

where

$$\tilde{\mathcal{M}}_n = \tilde{\mathcal{M}}_{\alpha}n_{\alpha}\,, \qquad \tilde{\mathcal{M}}_{\tau} = \tilde{\mathcal{M}}_{\alpha}\tau_{\alpha}\,, \qquad \tilde{\mathcal{M}}_{\alpha} = \frac{d^3}{12}\,(r_{\alpha}^{+} + r_{\alpha}^{-})\,.$$

Having found the virtual work equation $\delta W = \delta L$ one can derive the equations of equilibrium,

$$-N^{\alpha\beta},_{\beta} = 0, \qquad -Q^{\alpha},_{\alpha} - \underline{\mathcal{M}^{\alpha\beta},_{\alpha\beta}} = p\,,$$
$$-\mathfrak{M}^{\alpha\beta},_{\beta} + Q^{\alpha} = 0\,, \tag{6.6}$$

and natural boundary conditions along γ_σ,

$$N_n = \tilde{N}_n\,, \qquad N_\tau = \tilde{N}_\tau\,,$$
$$Q^{\alpha}n_{\alpha} + \mathcal{M}^{\alpha\beta},_{\beta}n_{\alpha} + \mathcal{M}_{\tau},_{\tau} = Q + \tilde{\mathcal{M}}_{\tau},_{\tau}\,, \tag{6.7}$$
$$\mathcal{M}_n = \tilde{\mathcal{M}}_n\,, \qquad \mathfrak{M}_n = \tilde{\mathfrak{M}}_n\,, \qquad \mathfrak{M}_\tau = \tilde{\mathfrak{M}}_\tau\,,$$

where $\mathcal{M}_n = \mathcal{M}^{\alpha\beta}n_{\alpha}n_{\beta}$, $\mathcal{M}_{\tau} = \mathcal{M}^{\alpha\beta}n_{\alpha}\tau_{\beta}$.

The equations (6.6) can be expressed in terms of kinematic fields. In the case when the face-plates are isotropic, see Sec. 4.1, these equations read

$$-2db^2\mu\Big(\Delta\varphi_{\alpha} + \frac{1+\nu}{1-\nu}\,\varphi_{\sigma},_{\sigma\alpha}\Big) + 2\,\frac{b^2}{c}\,\mu_c(\varphi_{\alpha} + w,_{\alpha}) = 0\,,$$
$$\underline{\frac{d^3\mu}{3(1-\nu)}}\,\Delta^2 w - 2\,\frac{b^2}{c}\,\mu_c(\Delta w + \operatorname{div}\boldsymbol{\varphi}) = p\,, \tag{6.8}$$

while the equations for u_{α}^0 have been given by (4.13). Moreover, one can note that (4.15) holds good here; $\chi = \gamma_1,_2 - \gamma_2,_1 = \varphi_1,_2 - \varphi_2,_1$.

REMARK 1. In the theory of sandwich plates proposed by Mindlin [11] the in-plane work of stresses in the core is additionally considered. If one neglects this work one obtains the model equivalent to the model of Hoff. Between Mindlin's kinematic and stress fields v, ψ_{α}, Q_{α}, $\mathcal{N}_{\alpha\beta}$, $\mathcal{M}_{\alpha\beta}$ and the entities used in the present paper the following relations hold:

$$v = w, \qquad \psi_\alpha = \left(\frac{b}{c} - 1\right) w,_\alpha + \frac{b}{c}\,\varphi_\alpha\,,$$

$$Q_\alpha = \frac{c}{b}\,Q^\alpha, \qquad \mathcal{N}_{\alpha\beta} = \frac{1}{b}\,\mathfrak{M}^{\alpha\beta}, \qquad \mathcal{M}_{\alpha\beta} = \mathcal{M}^{\alpha\beta}\,.$$

Thus $\psi_\alpha + w,_\alpha = \beta_\alpha$, standing for shear deformations of the core, see $(3.7)_2$ and (4.5). However, Mindlin's choice of unknowns makes a direct passage to the simplified model of Reissner [18] impossible.

6.3. Variational formulation of boundary-value problems

Change of the unknowns from $\mathbf{u}^0$, $\boldsymbol{\gamma}$, w to $\mathbf{u}^0$, $\boldsymbol{\varphi}$, w does not violate the $W(\Omega)$-ellipticity of the problem of the clamped plate. One can show that the bilinear form associated with the virtual work (6.3),

$$\begin{aligned} a_H(\mathbf{u}^0, \boldsymbol{\varphi}, w; \mathbf{v}^0, \boldsymbol{\psi}, v) = \int_\Omega \Big[& 2dA^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{u}^0)\gamma_{\lambda\mu}(\mathbf{v}^0) \\ & + \frac{d^3}{6}\,A^{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(w)\kappa_{\lambda\mu}(v) + 2db^2A^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\boldsymbol{\varphi})\gamma_{\lambda\mu}(\boldsymbol{\psi}) \\ & + 2\,\frac{b^2}{c}\,\mu_c(w,_\alpha + \varphi_\alpha)(v,_\alpha + \psi_\alpha)\Big]\,\mathrm{d}x\,, \end{aligned} \tag{6.9}$$

is symmetric and $W(\Omega)$-elliptic. Thus the problem of the clamped plate,

$$(\mathcal{P}_3)\begin{cases} \text{find } \mathbf{z} = (\mathbf{u}^0, \boldsymbol{\varphi}, w) \in W(\Omega) \quad \text{such that} \\ a_H(\mathbf{z}; \mathbf{v}) = f(v) \quad \text{for every } \mathbf{v} = (\mathbf{v}^0, \boldsymbol{\psi}, v) \in W(\Omega)\,, \end{cases}$$

is correctly posed. Other boundary-value problems admissible within the framework of this theory can be formulated by appropriate change of the space $W(\Omega)$ and the linear form $f(.)$.

REMARK 2. Let us note that the bilinear form $a_H(.;.)$ is similar but not identical with the bilinear form of Reddy [15], see [2, 9],

$$\begin{aligned} a_{JR}(\mathbf{u}^0, \boldsymbol{\varphi}, w; \mathbf{v}^0, \boldsymbol{\psi}, v) = \int_\Omega \Big\{ & 2hA^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{u}^0)\gamma_{\lambda\mu}(\mathbf{v}^0) \\ & + \frac{2h^3}{3}\,A^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\boldsymbol{\varphi})\gamma_{\lambda\mu}(\boldsymbol{\psi}) + \frac{5}{3}\,hC^{\alpha 3\beta 3}(w,_\alpha + \varphi_\alpha)(v,_\beta + \psi_\beta) \\ & + \frac{h^3}{126}\,A^{\alpha\beta\lambda\mu}[\gamma_{\alpha\beta}(\boldsymbol{\varphi}) - \kappa_{\alpha\beta}(w)][\gamma_{\lambda\mu}(\boldsymbol{\psi}) - \kappa_{\lambda\mu}(v)]\Big\}\,\mathrm{d}x\,. \end{aligned} \tag{6.10}$$

The above form refers to the case when the plate is transversely homogeneous. The nonlinear terms have been omitted.

In both theories of Hoff and Reissner the unknowns φ_α can be replaced by any combination $a\varphi_\alpha + bw,_\alpha$ and both theories will remain correctly posed. We shall see later that such a flexibility does not hold in the theory of Reissner [18].

7. On the Reissner [18] theory of sandwich plates

7.1. Passage to the model of Reissner

In the simplest possible model of sandwich plates with soft core the total energy is a sum of the energy of the in-plane deformation of face-plates, of the resultant bending deformation and of the shearing deformation of the core, see Reissner [18]; see also [6, 11, 14, 20]. This model can be arrived at from the model of Hoff by neglecting

(i) the bending energy $\mathcal{M}^{\alpha\beta}k_{\alpha\beta}$ stored in the face-plates,
(ii) the work done by the couple resultants $\tilde{\mathcal{M}}_{\alpha}$.

Thus in the model of Reissner the terms underlined in the expressions (6.3), (6.5) and in the equations of equilibrium (6.6), (6.8) are omitted. The number of natural boundary conditions is reduced from six to five. These conditions read

$$N_n = \tilde{N}_n, \qquad N_\tau = \tilde{N}_\tau, \qquad Q^\alpha n_\alpha = Q, \qquad \mathfrak{M}_n = \tilde{\mathfrak{M}}_n, \qquad \mathfrak{M}_\tau = \tilde{\mathfrak{M}}_\tau . \tag{7.1}$$

REMARK 3. By virtue of assuming the quantities φ_α as unknowns in the model of Hoff the passage to the model of Reissner is based upon only two simplifications (i), (ii). In the paper of Mindlin [11] this passage could have been shown only under an additional approximation $c\psi_\alpha - (d/2)w,_\alpha \approx c\psi_\alpha$, see Remark 1, which has led to the incorrect formula $2c\mu_c$ for the shearing stiffness. However, its value can be amended by introducing a shear-correction factor.

7.2. Well-posedness of the boundary-value problems expressed in terms of $(\mathbf{u}^0, \boldsymbol{\varphi}, w)$

Let Ω satisfy the regularity conditions assumed in Sec 4.2. For the sake of brevity let us suppose that the plate is clamped along $\Gamma_u = \Gamma_0$, viz. $\gamma_u = \gamma$. The space of kinematically admissible fields $(\mathbf{u}^0, \boldsymbol{\varphi}, w)$ is

$$U(\Omega) = [H_0^1(\Omega)]^2 \times [H_0^1(\Omega)]^2 \times H_0^1(\Omega) .$$

The bilinear form of the Reissner [18] model,

$$a_R(\mathbf{u}^0, \boldsymbol{\varphi}, w; \mathbf{v}^0, \boldsymbol{\psi}, v) = \int_\Omega \Big[2dA^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{u}^0)\gamma_{\lambda\mu}(\mathbf{v}^0) + 2db^2A^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\boldsymbol{\varphi})\gamma_{\lambda\mu}(\boldsymbol{\psi}) + 2\,\frac{b^2}{c}\,\mu_c(\varphi_\alpha + w,_\alpha)(\psi_\alpha + v,_\alpha)\Big]\,\mathrm{d}x , \tag{7.2}$$

is symmetric, continuous and $U(\Omega)$-elliptic. Continuity follows from relation $(4.17)_1$. The outline of the proof of ellipticity is as follows. According to the inequalities of Friedrichs and Korn there exist constants c_1, c_2 such that

$$\int_\Omega w,_\alpha w,_\alpha \,\mathrm{d}x \geqslant c_1 \|w\|^2_{H_0^1(\Omega)} , \tag{7.3}$$

$$\int_\Omega \gamma_{\alpha\beta}(\boldsymbol{\varphi})\gamma_{\alpha\beta}(\boldsymbol{\varphi})\,\mathrm{d}x \geqslant c_2 \|\boldsymbol{\varphi}\|^2_{[H_0^1(\Omega)]^2} . \tag{7.4}$$

For every $\beta > 0$ one can estimate

$$J = \int_\Omega (\varphi_\alpha + w,_\alpha)(\varphi_\alpha + w,_\alpha)\, dx \geq \frac{\beta}{1+\beta} \int_\Omega w,_\alpha w,_\alpha \, dx - \beta \int_\Omega \varphi_\alpha \varphi_\alpha \, dx ,$$

and by virtue of (7.3) one has

$$J \geq \frac{\beta c_1}{1+\beta} \|w\|^2_{H^1_0(\Omega)} - \beta \|\boldsymbol{\varphi}\|^2 , \tag{7.5}$$

where $\|.\| = \|.\|_{L^2(\Omega)}$. By using $(4.17)_2$ and the inequality (7.4) one obtains

$$\begin{aligned} &\int_\Omega 2db^2 A^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}(\boldsymbol{\varphi}) \gamma_{\lambda\mu}(\boldsymbol{\varphi})\, dx \geq 2db^2 k_2 c_2 \|\boldsymbol{\varphi}\|^2_{[H^1_0(\Omega)]^2} , \\ &\int_\Omega 2dA^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}(\mathbf{u}^0) \gamma_{\lambda\mu}(\mathbf{u}^0)\, dx \geq 2dk_2 c_2 \|\mathbf{u}^0\|^2_{[H^1_0(\Omega)]^2} . \end{aligned} \tag{7.6}$$

For $\beta < dck_2c_2/\mu_c$ there exists a constant $c_3 > 0$ such that

$$\begin{aligned} a_R(\mathbf{z}; \mathbf{z}) &\geq c_3[\|\mathbf{u}^0\|^2_{[H^1_0(\Omega)]^2} + \|\boldsymbol{\varphi}\|^2_{[H^1_0(\Omega)]^2} + \|w\|^2_{H^1_0(\Omega)}] \\ &= c_3 \|\mathbf{z}\|^2_{U(\Omega)} , \end{aligned} \tag{7.7}$$

where $\mathbf{z} = (\mathbf{u}^0, \boldsymbol{\varphi}, w)$, which confirms that the bilinear form (7.2) is $U(\Omega)$-elliptic. Let it be emphasized here that the above consideration is correct provided that all entities are non-dimensional. Thus, the following problem,

$$(\mathcal{P}_4) \begin{cases} \text{find } \mathbf{z} = (\mathbf{u}^0, \boldsymbol{\varphi}, w) \in U(\Omega) \quad \text{such that} \\ a_R(\mathbf{z}; \mathbf{v}) = f(v) \quad \text{for every } \mathbf{v} = (\mathbf{v}^0, \boldsymbol{\psi}, v) \in U(\Omega) , \end{cases}$$

is correctly posed.

7.3. Non-correctness of the boundary-value problems expressed in terms of $\mathbf{u}^0$, $\boldsymbol{\gamma}$, w

Let $\mathbf{z} = (\mathbf{u}^0, \boldsymbol{\gamma}, w)$ and $\mathbf{v} = (\mathbf{v}^0, \boldsymbol{\delta}, v)$ belong to the space $W(\Omega)$, see Sec. 5.2. We define the bilinear form $b_R(.;.)$ by

$$b_R(\mathbf{z}; \mathbf{v}) = a_R(\mathbf{u}^0, \boldsymbol{\varphi}, w; \mathbf{v}^0, \boldsymbol{\delta}, v) . \tag{7.8}$$

Hence

$$\begin{aligned} b_R(\mathbf{z}; \mathbf{v}) = \int_\Omega \Big\{ &2dA^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}(\mathbf{u}^0) \gamma_{\lambda\mu}(\mathbf{v}^0) + 2\,\frac{b^2}{c}\, \mu_c \gamma_\alpha \delta_\alpha \\ &+ 2db^2 A^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(\boldsymbol{\gamma}) + \kappa_{\alpha\beta}(w)][\gamma_{\lambda\mu}(\boldsymbol{\delta}) + \kappa_{\lambda\mu}(v)] \Big\}\, dx . \end{aligned} \tag{7.9}$$

Prior to showing that $b_R(.;.)$ is not $W(\Omega)$-elliptic let us refer to

LEMMA 2. *Let Ω satisfy the regularity conditions assumed in Sec. 4.2. Then for an arbitrary constant $M > 0$ there exists $w \in H_0^2(\Omega)$ such that*

$$\int_\Omega w,_\alpha w,_\alpha \, dx < M \int_\Omega (w^2 + 2w,_\alpha w,_\alpha + 2w,_{\alpha\beta} w,_{\alpha\beta}) \, dx . \tag{7.10}$$

As it is standard, the proof of this lemma will be omitted.

Having recalled the inequality (7.10) one can show that for every $M_1 > 0$ there exists $\mathbf{z} = (\mathbf{u}^0, \boldsymbol{\gamma}, w) \in W(\Omega)$ such that

$$b_R(\mathbf{z}; \mathbf{z}) < M_1 \|\mathbf{z}\|^2_{W(\Omega)} . \tag{7.11}$$

Let us fix $M_1 > 0$ and choose $\mathbf{z} = (\mathbf{0}, \nabla w, w)$, $w \in H_0^2(\Omega)$. Then

$$b_R(\mathbf{z}; \mathbf{z}) = 2 \frac{b^2}{c} \mu_c \int_\Omega w,_\alpha w,_\alpha \, dx .$$

According to Lemma 2 for $M = M_1 c / 2b^2 \mu_c$ there exists $w \in H_0^2(\Omega)$ such that

$$\begin{aligned} b_R(\mathbf{z}; \mathbf{z}) &< 2 \frac{b^2}{c} \mu_c M \int_\Omega (w^2 + 2w,_\alpha w,_\alpha + 2w,_{\alpha\beta} w,_{\alpha\beta}) \, dx \\ &= M_1 \left[\int_\Omega (w^2 + w,_\alpha w,_\alpha + w,_{\alpha\beta} w,_{\alpha\beta}) \, dx + \int_\Omega (\gamma_\alpha \gamma_\alpha + \gamma_{\alpha},_{\beta} \gamma_{\alpha},_{\beta}) \, dx \right] \\ &= M_1 \|\mathbf{z}\|^2_{W(\Omega)} , \end{aligned}$$

which confirms that the bilinear form is not $W(\Omega)$-elliptic.

8. Concluding remarks

It has been shown that the model of Hoff can be formulated in terms of the fields $\mathbf{u}^0$, $\boldsymbol{\psi} = a\boldsymbol{\varphi} + b\nabla w$, and w; a, b being arbitrary constants. Then we have proved that neglecting the bending stiffnesses of face-plates deprives the formulation of the model of such flexibility. It occurs that Reissner's model should be cast in terms of $\mathbf{u}^0$, $\boldsymbol{\varphi}$, w. This choice is physically correct because φ_α stand for the total angles of rotation of fibres lying on the middle plane. Moreover, this is the only choice that assures ellipticity of the bilinear form $a_R(.;.)$. In other words, if we assume $\boldsymbol{\psi} = a\boldsymbol{\varphi} + b\nabla w$ as unknowns that replace $\boldsymbol{\varphi}$, then the condition of ellipticity stipulates that $b = 0$. This remark refers not only to the theory of sandwich plates but to all versions of the Reissner [17] theory of homogeneous plates with moderate thickness, in particular to those discussed in the recent review paper by Reddy [16]. Therefore, let it be emphasized here that the novelty of a theory of plates cannot be appreciated until the relevant bilinear form is found and compared with that of Reissner.

Both the Hoff and Reissner models are elliptic. However, note that the constant k_4 in (5.9) depends upon the ratio d/b while the constant c_3 in (7.7) depends upon dc. After introducing non-dimensional coordinates the latter constant would depend on dc/l^2, l being a diameter of Ω. Thus the constant c_3 depends on the relative thickness of the plate. This dependence makes the straightforward application of the finite-element method impossible, and special rearrangements are necessary, cf. Brezzi and Fortin [3].

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